Modeling An Apollonian Cone

A journey not a destination

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19:55 CDST

The author makes no claim of expertise in this subject area. Indeed he has none. The artifacts found here were developed after being introduced to Apollonius' Conics in a quest to answer the question, "What is a parabola?" Except for Ty's occasional reminders that "it's a journey not a destination" the quest likely would have been abandoned early on. Learning is a never ending journey though and there is no fear that these artifacts will reveal the story's ending prematurely. It is hoped, however, that they will be of some worth to a reader choosing to travel this way.

A youth who had begun to read geometry with Euclid, when he had learnt the first proposition, inquired, "What do I get by learning these things?" So Euclid called a slave and said "Give him three pence, since he must make a gain out of what he learns."
Stobaeus, Extracts

"If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker."
Albert Einstein

"Modern writers employ fractions instead of ratios, and with great advantage. But the student who leaves untouched that consideration of ratio which includes incommensurables as well as commensurables will never be more than a mathematician to a certain number of decimal places."
Penny Cyclopaedia of the Society for the Diffusion of Useful Knowledge, 1821 edition.

Introduction

In Euclid's[1]era geometers constructed cones by rotating a right triangle about one of it's perpendicular sides. The side became the axis of the cone positioning the apex over the center of the circular base. Conic sections came from cutting the cone with a plane that was perpendicular to a side of the cone. They were characterized as being a section of an acute, right, or obtuse cone as the apex angle of the cone was acute, right or obtuse. In today's parlance these cones are right circular cones.
Rather than starting from a triangle, Apollonius[2]began construction of a cone with a circle and a point not on the plane of the circle. From the point he drew a line to the circle, extended it infinitely in both directions, and then rotated the line about the circle to form a conic surface. He called the point the apex, the circle the base, and the portion of the conic surface between them the cone. The line from the apex to the center of the base he defined to be the cone axis. The only restriction placed on the cutting plane was that it intersect the base plane in a line that is perpendicular to a diameter of the base. These cones are characterized today as oblique circular cones.

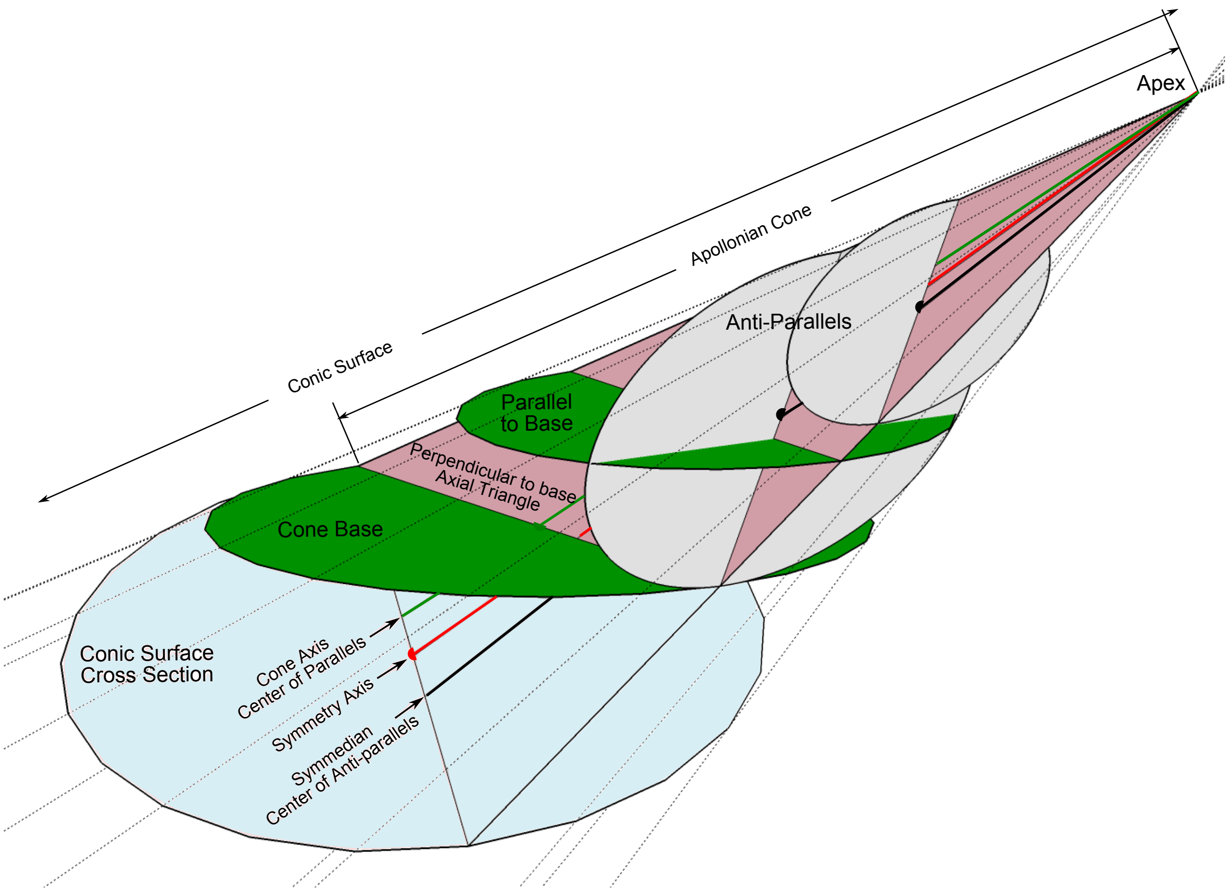


Figure 1: Oblique Cone Symmedians

The names "right circular" and "'oblique circular" imply a similarity between the cones that does does not exist. They are similar in that circular sections are created when either cone is cut by planes parallel to their base and an Euclid cone is the result when an Apollonian cone is created with the apex point positioned over the center of the base circle. Beyond that, however, an Apollonian cone has an elliptical cross section and, except in that one instance, the similarities are special instances in Apollonius' generalization. It is interesting to note that some sixteen centuries latter Desargues describes Apollonius' work as being special cases of his more general projective geometry extensions. Two aspects of Apollonius' extension that are seldom encountered in today's Cartesian based geometry are non-orthogonal conjugate diameters and anti-parallels. Both involve concepts more likely to be encountered in framing construction and air handling duct work than in modern presentations of mathematics.

The most readily identifiable and quantifiable parameters describing an intersection of a plane with an oblique cone are conjugate diameters rather than the axes commonly used when working with right cones. This difference was the primary reason for undertaking this study.
The cone axis passes through the center of the base and the circular sections parallel to it. In a right cone the cone axis is co-linear with the symmetry axis and the base circle reflects onto itself. This is not the case in an oblique cone. The oblique cone axis does not lie along the symmetry axis which bi-sects the apex angle of the cone. Consequential the circles parallel to the base are reflected into another set of circles, called the anti-parallels, whose centers lie on the symmedian to the cone axis. Figure 1 shows the relationship between these axes, the infinite elliptical conic surface which has a rotational symmetry of two and the Apollonian oblique cone. The constructor lines of the cone connect in pairs to opposite ends of base diameters forming axial triangles. The cone axis is the line of intersection of these planes and hence it lies in the planes of all axial triangles. On the other hand the symmetry axis and the symmedians all lie together in the plane of the triangle formed from the shortest and longest constructors. This is the only axial triangle that is always perpendicular to the base.
Prior to this study we had worked through some propositions in Apollonius' treatise on Conics using graphical constructions. It was not even close to being a compass and straight edge approach, but ratios were treated as ratios and evaluated via a graphical construction rather than treating ratios as fractions and evaluating them numerically. On the other hand a protractor was used to erect perpendiculars and a ruler might be used to find the mid point of a line. Both tasks that are easily accomplished using a compass but get messy when the drawing program being used only draws filled circles.
The propositions of interest were those that dealt with creating an elliptical section having non orthogonal conjugate diameters in an oblique cone and then recreating it in a right cone. Specifically propositions I.13, 1.56 - 58, and VI.30 of the Conics. In this study some mathematical models of the geometric constructions used in the earlier study are developed.

Supplemental Example Workbooks

If the workbook files are kept in the same directory as the main.pdf file a workbooks can, in many instances, be opened by simply clicking the file reference. If that doesn't work for you the workbooks can be opened directly in MS Excel© using the workbook name. Recent versions of Libre OfficeI© and SymphonyI© also seem to work but the scale and colors of plots are more apt to need fine tuning. These spreadsheet workbooks are based on equations derived in this report and were developed to verify and further our understanding of cone/plane intersections. They are included in the hope that they may have some educational worth to others interested in Apollonius' conics.. No warranty of any type nor fitness of use for any purpose is expressed or implied. Use at your own risk.
Professionally written software that uses Apollonius' method to design patterns for sheet metal workers is available in the Conet program found at: `http://www.tyharness.co.uk/cones/conemath.htm`. Educators interested in an educational version of the program should contact Ty through the email address on the web site.

BasicConicCalculations Demo
deWitt Demo
AxesTaylorP299 Demo
AxesPGeo49 Demo
ComboNoMacro Demo
AxesMacroCompatMode Demo

[1] In Book XI of *Euclid's Elements*, `http://aleph0.clarku.edu/~djoyce/java/elements/bookXI/bookXI.html`, a cone of Euclid's day is described thusly.
Def. 18. When a right triangle with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cone. And, if the straight line which remains fixed equals the remaining side about the right angle which is carried round, the cone will be right-angled; if less, obtuse-angled; and if greater, acute-angled.
Def. 19. The axis of the cone is the straight line which remains fixed and about which the triangle is turned.
Def. 20. And the base is the circle described by the straight line which is carried round.

[2] T. L. Heath's *Treatise on Conic Sections* is available from `http://www.wilbourhall.org/pdfs/Treatise_on_Conic_Sections.pdf`

Calculating Effective Angles

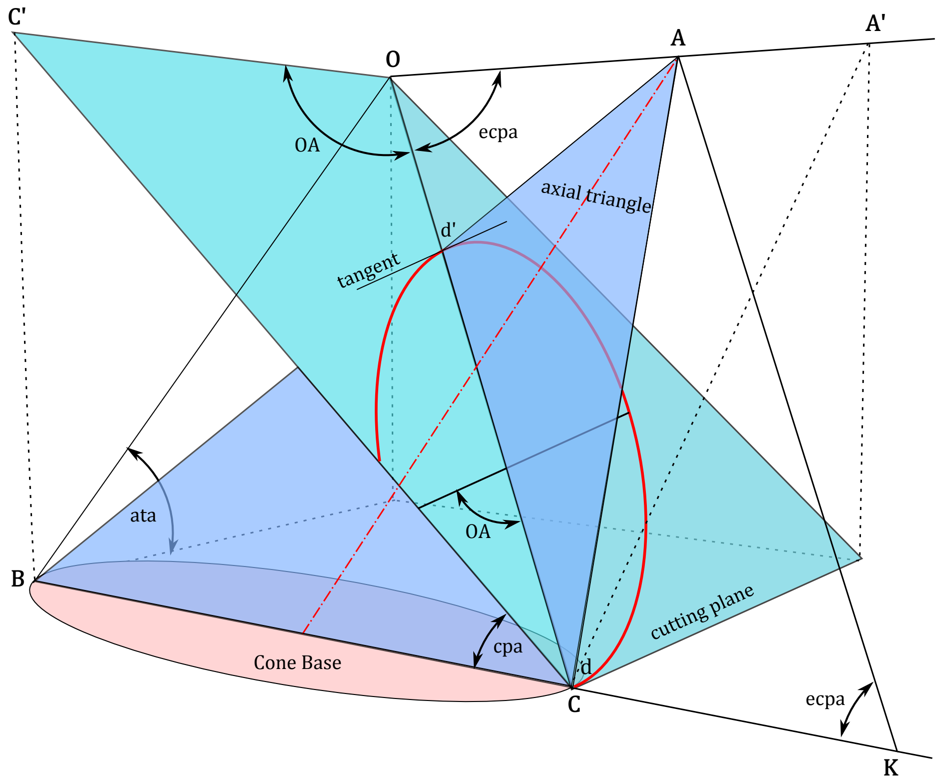


Figure 2: Cone and Plane Intersection

The conic sections created by intersecting a cone with a plane lie on the cutting plane. One diameter of the cone base will be perpendicular to the line of intersection between the cutting plane and the base plane. Lines from the cone apex to opposite ends of this diameter form the axial triangle. In Apollonius' terms, the principal diameter of the section, dd', lies along the line of intersection between the cutting plane and the imaginary axial triangle plane. In a right circular cone it will be the longest diameter but in an oblique cone it will only be the longest when the cone is orientated such that the axial triangle is perpendicular to the base or the section is a circle.
The tangents to the section at the ends of the principal axis are parallel to the line of intersection between the cutting and base planes. The mid-point of the principal diameter is at the center of the section. It passes through the mid-points of all the chords that are parallel to the these tangents which are collectively known as ordinates. The ordinate passing through the center of the section will thus also be a diameter and is called the conjugate diameter. The two diameters form a conjugate pair and the angle between them is the ordinate angle.
In a right circular cone the principal and conjugate diameters will be perpendicular but in an oblique cone the ordinate angle, OA, will be right only when the section is a circle or the axial triangle is perpendicular to the base plane. In all other instances the ordinate angle varies with the slope of both the cutting plane and axial triangle.
Construction workers encounter the same phenomena when framing roofs where two roofs intersect and the rafters for "hips" and "valleys" must be cut at compound angles to fit properly. It may also be encountered by a home owner installing molding with mitered joints or by a hobbyist constructing mitered wooden boxes. This on-line Compound Joint Angle Calculator Click Here or, from http://www.pdxtex.com/canoe/compound.htm, discusses the problem and can be used to calculate the angles involved for simple joinery.

Definitions of symbols used in Figure 2.

BC - axial triangle base
AK - line through A parallel to the principal diameter of the conic and intersecting the extension of BC at K.
ata - axial triangle angle - slope of the axial triangle plane
cpa - cutting plane angle - slope of cutting plane
ecpa - effective cutting plane angle - slope of cutting plane
OA - ordinate angle - angle between principal diameter and ordinates

The ordinates are always parallel to line of intersection between the cutting and base planes. However, OA is only valid on the inclined cutting plane. It would not, for instance, correctly indicate the projection of the principal diameter direction onto the base plane.

$$CC' = BC'/SIN(cpa)$$
$$OC' = BC'/TAN(ata)$$
$$TAN(OA) = CC'/OC' = TAN(ata)/SIN(cpa)$$
$$\mathbf{OA = ATAN(TAN(OA)) = ATAN(TAN(ata)\ /\ SIN(cpa))} \tag{1}$$

$$CA' = BC'/SIN(ata)$$
$$OA' = BC'/TAN(cpa)$$
$$TAN(ecpa) = OA'/OC' = TAN(cpa)/SIN(ata)$$
$$\mathbf{ecpa = ATAN(TAN(ecpa)){=}ATAN(TAN(cpa)\ /\ SIN(ata))} \tag{2}$$

Calculating Section Characteristics

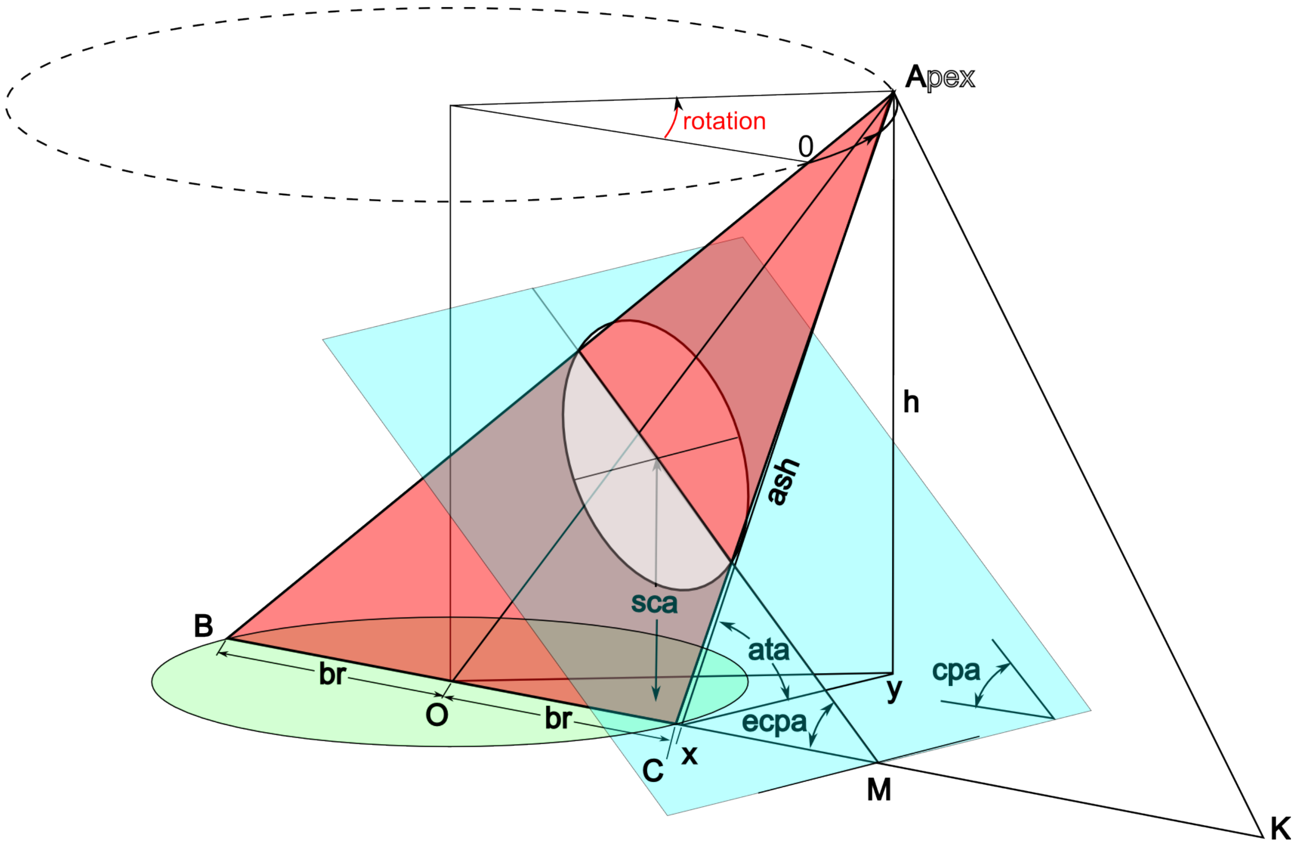


Figure 3: Cone Geometry

With the exception of the angle of the ordinates, Apollonius was able to uniquely characterize conic sections with a parameter p that could be constructed using four lines that all lie in the same plane.

How the physical properties of the cone affected these lines was not considered. Whereas, in this study, that is the primary interest. Figure 3 identifies the cone/cutting plane components of interest and we begin here to establish their relationship to the chateristics of the cone sections.

Explanation of symbols used in Figure 3.

Physical Properties:

br - cone base radius
h - cone height - perpendicular distance from base plane to cone apex
Apex - offset from center of base to the projection of the apex onto the base plane measured along an extension of the base of the perpendicular axial triangle.
rotation - rotation of cone about the base center measured CCW from its position when the cone and x axis are aligned.
CM - distance from cone base to intersection of axial triangle base extension with extension of section principal diameter.
sca - Section Center Altitude
These two are discussed in a following section.
ecpa - effective cutting plane angle
OA - ordinate angle

Calculated values:

$$x = Apex * COS(rotation) \qquad \text{– projection of apex offset onto x axis}$$
$$y = ABS(Apex * SIN(rotation)) \qquad \text{– projection of apex offset onto y axis}$$
$$ata = IF(y = 0, 90, ATAN(h/y)) \qquad \text{– slant angle of axial triangle plane}$$
$$ash = \frac{h}{SIN(ata)} \qquad \text{–slant height of axial triangle}$$
$$AK = \frac{ash}{SIN(ecpa)} \qquad \text{– Apollonius variable}$$
$$BK = \frac{ash}{TAN(ecpa)} + br + x \qquad \text{– Apollonius variable}$$
$$CK = BK - 2 * br \qquad -Apolloniusvariable$$

Calculating Conic Section Altitude and Diameter

The diameter of the conic section, dd', is dependent upon the characteristics of the cone and the altitude and angle at which it is intersected by the cutting plane. Apollonius showed that the characteristics of a cone with base diameter BC are captured in the the three lengths, AK, BK and CK where AK is drawn parallel to the cutting plane. He used the quotient $\frac{AK^2}{BK * CK}$ and the diameter dd' to characterize sections. The three lengths will be used here to derive formula for the diameter and section altitude using the distance from the cone base to the line of intersection between the cutting plane and the cone base plane, CM.

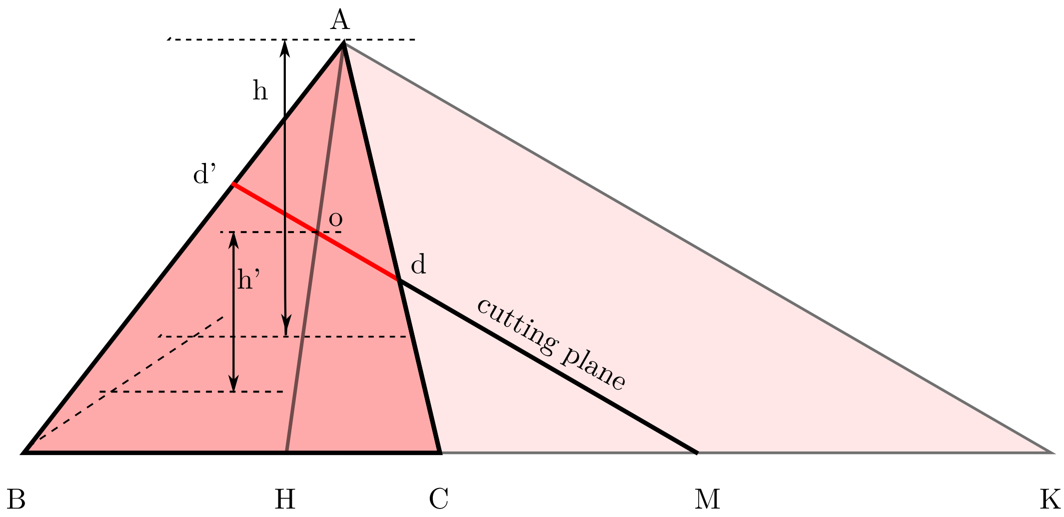


Figure 4: Axial Triangle Plane

Derivation of diameter dd' when CM is known:

$$(dd' + dM) : AK :: (BC + CM) : BK \tag{3}$$
$$dd' + dM = \frac{AK(BC + CM)}{BK} \tag{4}$$
$$dd' = \frac{AK(BC + CM)}{BK - dm} \tag{5}$$
$$dM : CM :: AK : CK \tag{6}$$
$$dM = \frac{CM * AK}{CK} \tag{7}$$
$$dd' = \frac{AK(BC + CM)}{BK} - \frac{CM * AK}{CK} \tag{8}$$
$$dd' = \frac{AK(BC * CK + CM(CK - BK))}{BK * CK} \tag{9}$$
$$dd' = \frac{AK(CK - CM)(BK - CK)}{BK * CK} \tag{10}$$
$$dd' = \frac{AK * BC * (CK - CM)}{BK * CK} \tag{11}$$

The axis of an oblique cone connects the Apex to the center of the base and passes through the center of all sections parallel to the base. However, the line HA passes through the mid-point of the diameter of the section shown but does not pass through the center of the base.
Derivation of the section altitude h' when CM and dd' are known:

$$h' : h :: Ho : HA = (dM + dd'/2) : AK = (2dM + dd')/2 : AK \tag{12}$$
$$h' = \frac{h}{2AK}(2dM + dd') \tag{13}$$
$$h' = \frac{h}{2AK}(\frac{2CM * AK}{CK} + dd') \tag{14}$$
$$h' = h(\frac{CM}{CK} + \frac{dd'}{2AK}) \tag{15}$$

Starting from equation 11 an equation can be derived for h' when dd' is not known:

$$h' = \frac{h}{2AK}(\frac{CM * AK}{CK} + \frac{AK(BC + CM))}{BK} \tag{16}$$

$$h' = \frac{h * AK(CM(BK + CK) + BC * CK)}{2AK * CK * BK} \tag{17}$$

$$h' = \frac{h(CM(BK + CK) + BC * CK)}{2BK * CK} \tag{18}$$

Finally, if h' is given, equation 16 can be re-factored to calculate CM which in turn can be used in equation 9 to calculate dd'.

$$CM = \frac{2h'(CK * BK) - h * BC * CK)}{h(BK + CK)} \tag{19}$$

Drawing an Ellipse from Conjugate Diameters

A method for drawing ellipse using the the conjugate diameters is shown in Jan de Witt's Elementa Curvarum Linearum, Liber Primus translated by A. W. Grootendorst and published by Springer. It can be used to draw ellipse when the diameters are orthogonal but being orthogonal is not a necessary condition. It can be used to draw sections using conjugate diameter pairs from the preceding calculations. De Witt's procedure is described thusly:

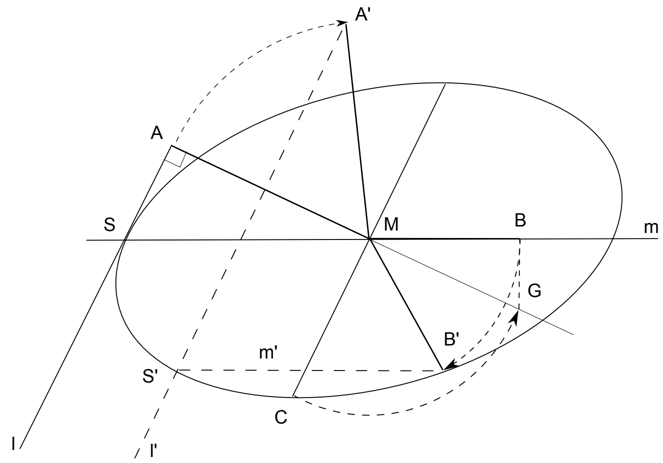


Figure 2.26

> Next the case is considered in which the angle AMB is not a straight angle (see Figure 2.26). In this case the curve is generated as follows: a straight line l is drawn through A perpendicular to MA intersecting the line m through B and M at S. Then the angle AMB rotates about M and in the new position $A'MB'$ of the angle AMB, the lines $l'$and m' are drawn through A and B parallel to l and m, respectively. The point of intersection of l' and m' is called S' and we are interested in the curve described by S', if AMB continuously rotates about the point M and if in every new position of the angle AMB lines are drawn parallel to l and m. This curve turns out to be an ellipse with M as its center and MS as a semidiameter. The corresponding second diameter is parallel to AS; its length is twice that of the segment MG, whereby G has been selected on the produced part of AM, in such a way that its projection on m coincides with the point B. Jan de Witt remarks that the curve described by S consists of two straight lines parallel to m if the angle AMB is a right angle.

The lengths of a pair of conjugate diameters and the angle between them are required for this construction. The general procedure is to create the angle AMB from the supplied data and then

tabulate the coordinates of S' as the angle is stepped around the center. The coordinates are calculated by finding the intersection of l' and m'.
The derivations of the needed equations are based on Figure 5 which shows the setup and drawing parameters. R, one side of the angle AMB, is the projection of MS onto the perpendicular to the ordinates AM. The other side of the angle, r, is constructed by marking off CM on the extension of AM (G) and then projecting it onto m at B. I chose to position the conjugate diameter horizontally and make the y axis lie along the principal diameter. The x axis is perpendicular to it. Using these axes, the equations of the lines intersecting lines that define the points of the curve are:

$$\text{line } m'$$
$$x = k = r * SIN(theta + alpha)$$
$$\text{line } l'$$
$$y = a * x + b$$

The parameters a and b can be determined by evaluating l' at A'.

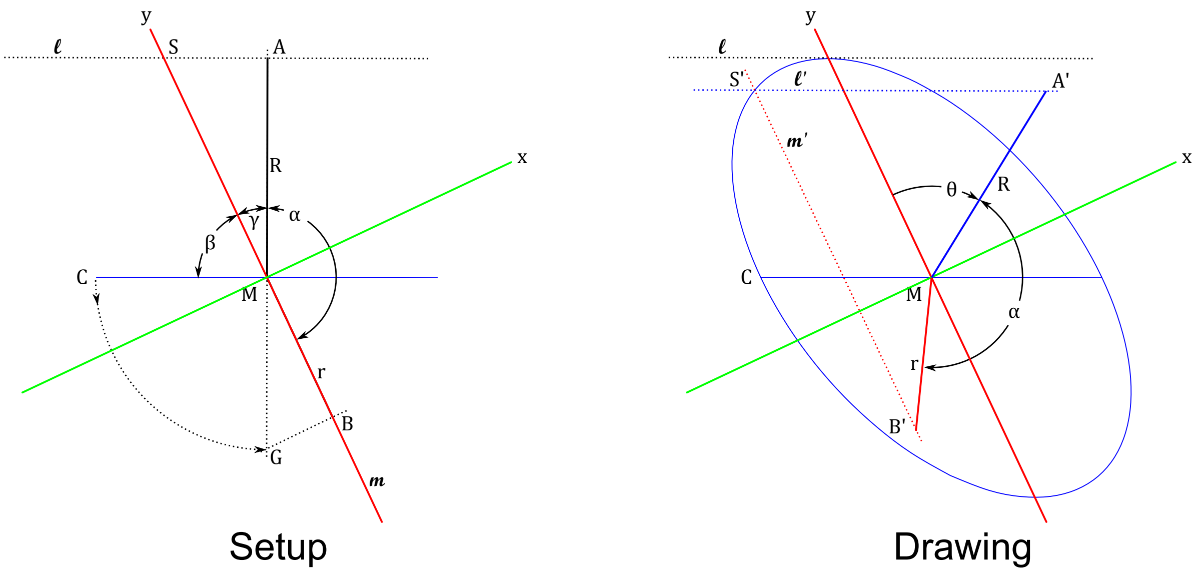


Figure 5: de Witt Set Up and Operation

$b = y - a * x = R * COS(theta) - a * R * SIN(theta)$
then at $x = 0, y = b = COS(theta) + R * SIN(theta)TAN(gamma) = R * COS(theta) - a * R * SIN(theta)$
From which $a = -TAN(gamma)$.
Hence the equations for the (x,y) coordinates of the point in the curve are:

$$x = r * SIN(theta + alpha)$$
$$y = a * x + b$$
$$where$$
$$a = -TAN(gamma)$$
$$b = R * COS(theta) - a * R * SIN(theta)$$

I wanted $theta = 0$ to always plot $(0, y)$ so I added an offset which can be set to gamma to accomplish this.

Finding The Axes of a Section

Apollonius characterized conic sections by the principal diameter and a parameter based on the physical characteristics of the cone/cutting plane configuration that produced the section. He defined the characteristics involved as $PP' : PL :: sq.AK^2 : Rect.BK, CK$ where PP' is the principal diameter and PL is the parameter. The parameter is sometimes referred to as p. AK, BK, CK were defined in an earlier part of this document. Frequently Apollonius attached the parameter perpendicular to one end of the principal diameter. In other instances the diameter and parameter appeared as a right triangle with sides PP' and PL. The equations developed earlier for calculating section diameters also included an equation for calculating the p ratio $\frac{BK * CK}{AK^2}$. The parameter p is the product of this ratio and the principal diameter.

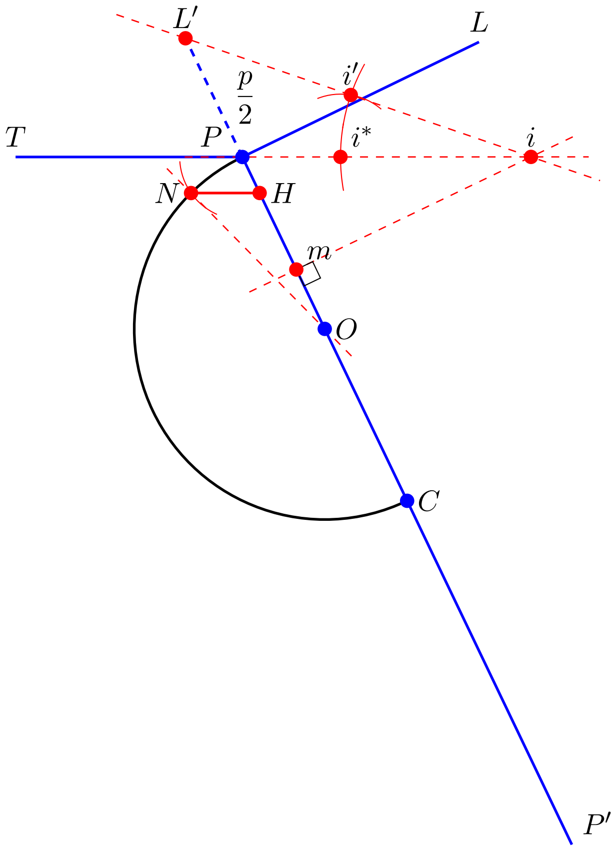
Proposition I.56 and VI.30 both describe construction of elliptical sections in a right cone and require that the major axis of the section be known. I.58 is Apollonius' method for finding the Major axis and its associated p value when the principal diameter and not the axis of a given curve are known. In I.58 the principal diameter, the parameter p, and the ordinate angle are assumed to be given. Apollonius does not reference the curve itself other than to say, after finding the major axis, that the curve can be drawn using the method of I.56.

Finding The Major Axis Using Principal Diameter, Parameter and Ordinate Angle

The proof begins by describing how these are to be laid out: a line of arbitrary length PT is drawn at the ordinate angle to a line PP' representing the principal diameter as shown in the figure at the right.

After laying out the "givens", Apollonius' proposition I.58 continues with the construction of an ordinate of a semi-circle that is done without comment. In the translations of the Conics an accompanying footnote references a method for the construction that is attributed to Eutocius. A presentation of Eutocius' method that is much clearer than those in the footnotes can be found in Colin McKinney's[3]2010 University of Iowa doctoral Thesis entitled "Conjugate Diameters: Apollonius of Perga and Eutocius of Ascalon." The diagrams and his discussion related to Book I, Propositions 55 and 58 are particularly helpful.

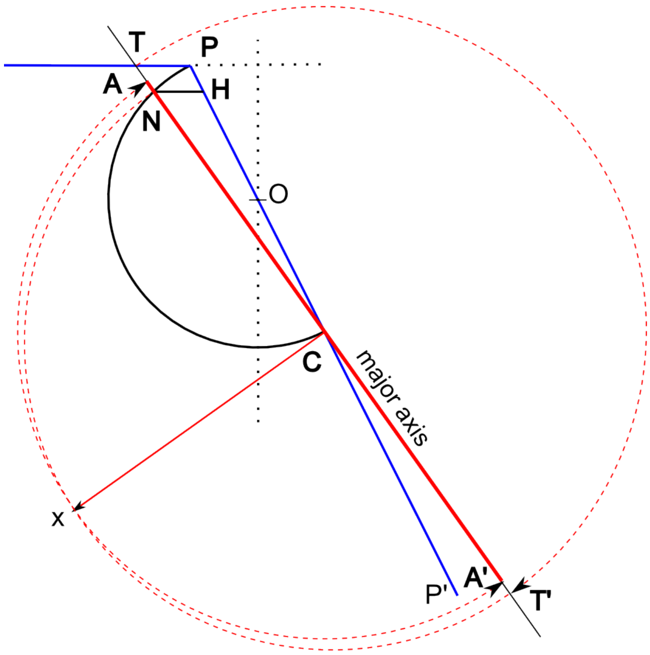
Referring to the I.58 Setup figure, the construction calls for a semicircle to be drawn on CP with center at O and the line NH to be drawn parallel to PT so that $sq.NH : rect.PH, HC :: p : PP'$. We present here a simple method of constructing NH that Apollonius might have used and considered to not need elaboration. Readers interested in Eutocius' method of construction are referred to McKinney's dissertation.



Apollonius' I.58 Setup

PL is the parameter p. Extend the line CP by the length $\frac{p}{2}$ to the point L'. Erect a perpendicular on CL' at m, the midpoint of CL', and extend it to intersect the extension of line TP at i. Draw the line $L'i$. Copy the angle i^*ii' to PON. Using a compass, from the point i, mark the length OP on iP at i^* and iL' at i'. From P mark the length i^*i' on the arc PC at N. With a drawing program it may be easier to copy the angle using a protractor to draw the line ON. Draw a line through N parallel to PT intersecting OP at H. NH is the required ordinate.

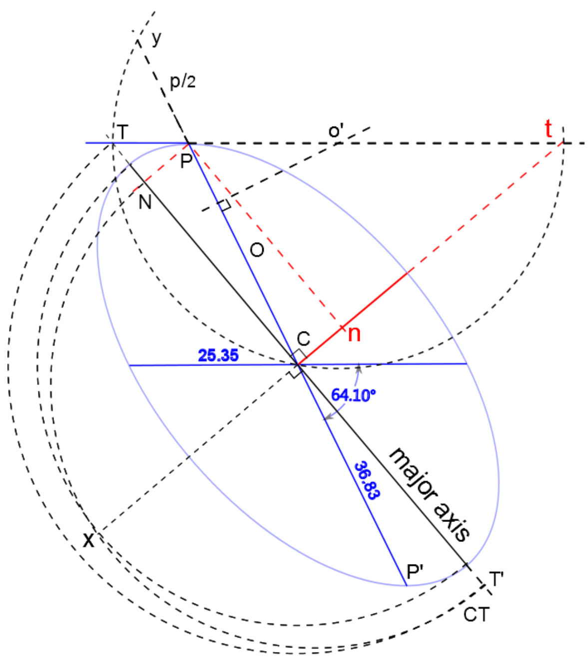
Once the point N is determined the line CN can be drawn and extended to intersect the line TP fixing the arbitrary point T. The major axis lies along CT.



Apollonius I.58 Construction of Major Axis

The length of the semi-axis is equal the mean proportional between CN and CT and can be constructed by drawing a semi-circle centered at C with a radius of CT and drawn to intersect at T' the extension of TC through C. Construct a second semi-circle on NCT' centered at the mid-point of NCT'. Erect a perpendicular on NCT' at C and extend it to intersect the second semi-circle NT' at x. Cx is the length of required semi-major axis and the major axis, AA', can be marked off on TT' with a circle or radius Cx centered at C. AA', shown in red, is the required major axis

A method that replaces the semi-circle on CP and the ordinate HN with a larger circle and parallel lines to establish the directions of the axes and lengths of the semi-axes is shown in this third drawing. The accompanying Spreadsheet implements a version of this method which is based on Problem 49 from Thomas Bradley's[4]1834 text *Practical Geometry, Linear Perspective and Projection.*



Apollonius I.58 Alternate

In itself this method does not find the parameter for the axes. However, it is compatible with the I.58 construction which can be used to do so. When the section is taller than it is wide, p will be less than the diameter. When it is wider than it is tall, p will be greater than the diameter and p will equal to the diameters when the conjugate diameters are equal.
This occurs when the cutting plane is parallel to the base or parallel to the anti-parallels. It can, however, also occur when the sections are not circles. In this latter case it indicates the cutting plane is inclined at the angle at which a transition between tall and narrow and short and wide sections occurs. Steeper angles will produce taller sections while lesser slopes will produce shorter sections. This phenomena does not occur when the cone is a right circular cone. Thus before a short and fat sections can be reconstructed in a right cone, Apollonius' I.57 method must be used to transform it to a tall and narrow orientation.

Using Conjugate Diameters and Ordinate Angle

One of the simplest methods we have found for finding axes is described in Charles Taylor's[5] *An Introduction to the Ancient and Modern Geometry on Conics*. Using a pair of conjugate diameters it finds the lengths and direction of the axes. Taylor's description of the method and example constructions are shown on the drawings made by TY Harness and included here. The examples show the method works both when $CP > CD$ and $CP < CD$. It is easily adapted for use in a spreadsheet and equations are derived and included for that purpose. Excel VBA code for implementing it as a Macro is given. I am told the macro works in Lotus Symphony and Libre Office 3.5 when it is entered in Excel and saved in Excel 2003 format.

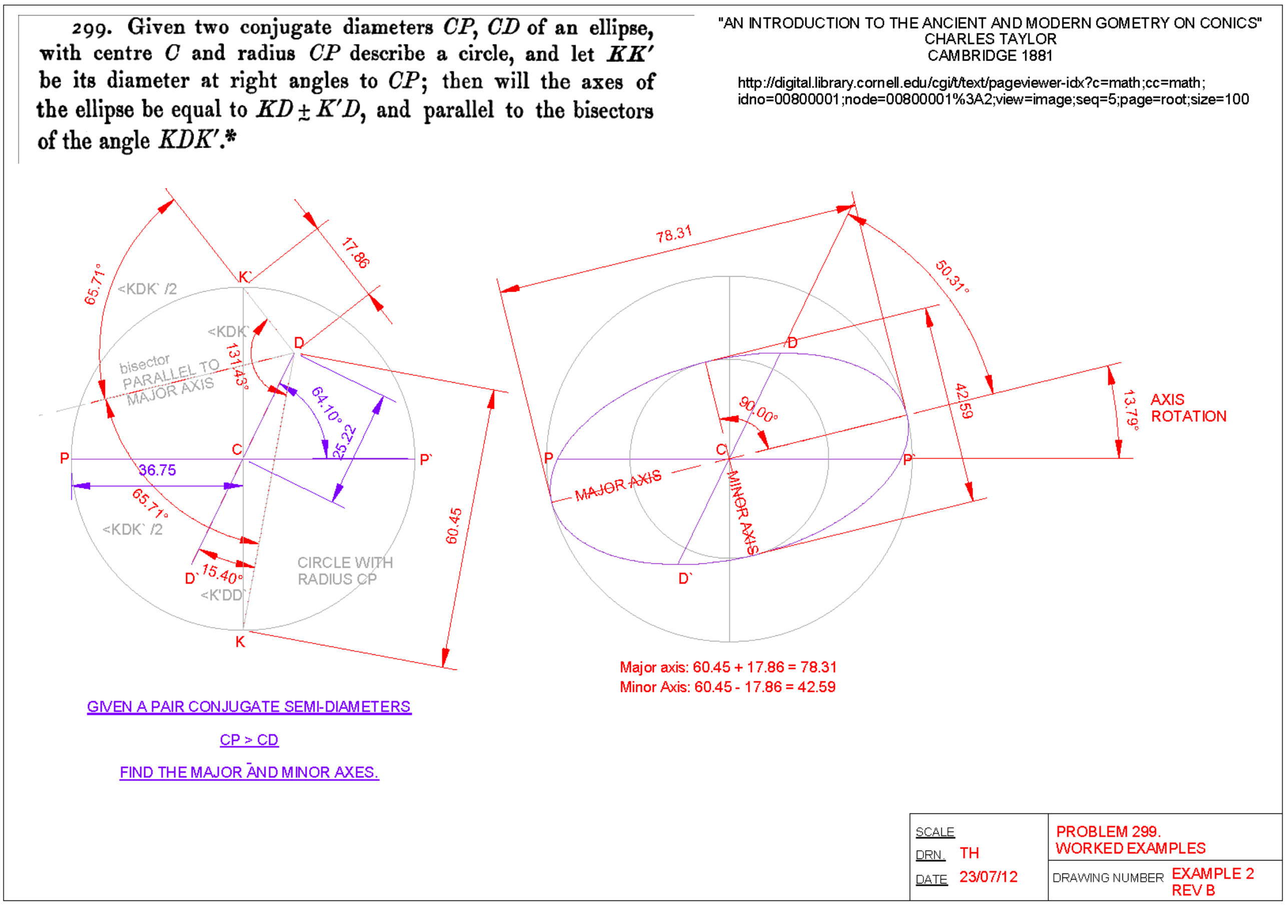


Figure 6: Taylor's Graphical Construction of Ellipse Axis when $CP > CD$

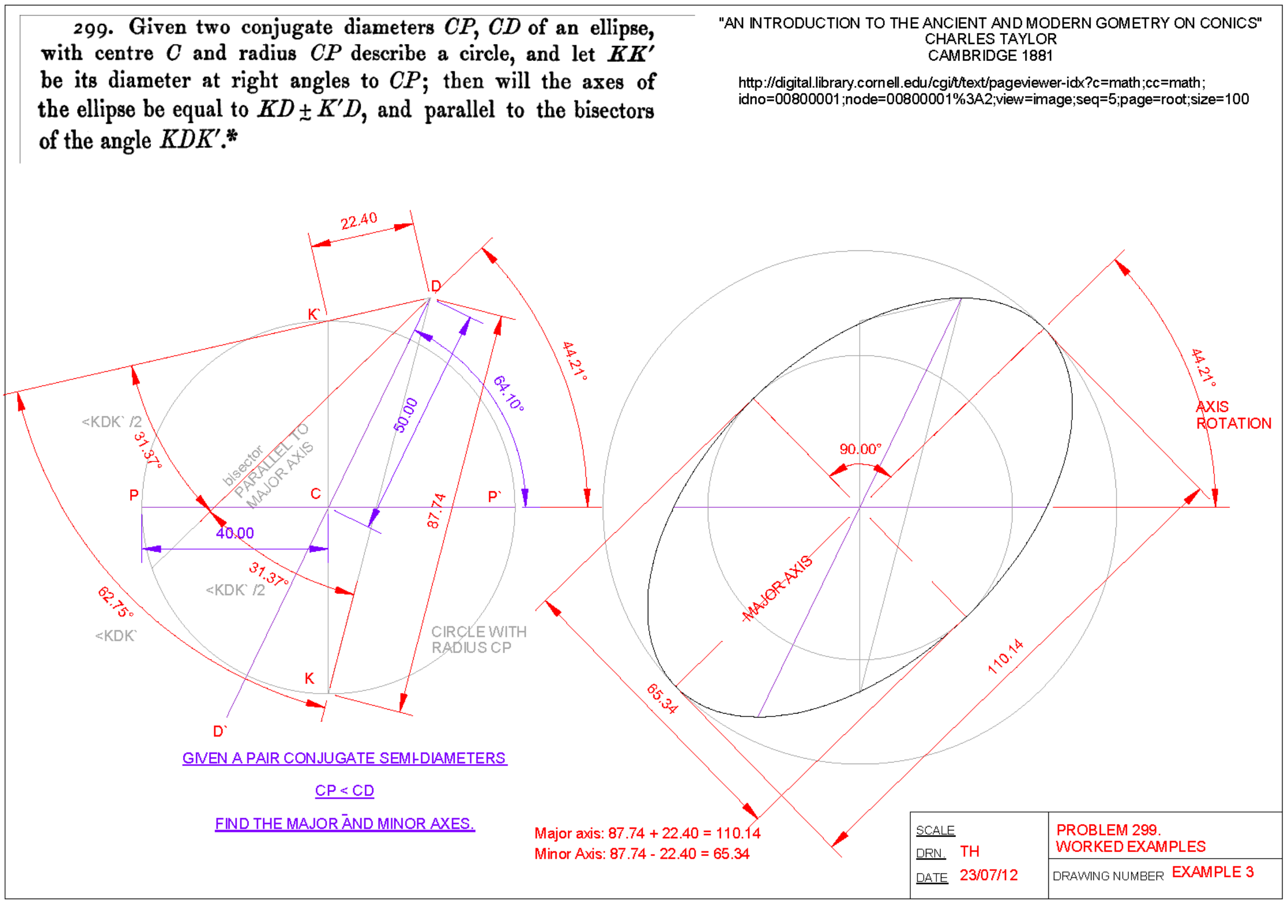


Figure 7: Taylor's Graphical Construction of Ellipse Axis when $CP < CD$

[3]Colin McKinney, doctoral Thesis *Conjugate Diameters: Apollonius of Perga and Eutocius of Ascalon*, University of Iowa 2010. Available from: http://ir.uiowa.edu/cgi/viewcontent.cgi?article=1896&context=etd

[4]Thomas Bradley, *Practical Geometry, Linear Perspective, and Projection*, 1934. Can be read as an ebook from Google Books at http://books.google.com/books/reader?id=FetJAAAAMAAJ&printsec=frontcover&output=reader&pg=GBS.PR1, or, downloaded in several formats from http://archive.org/details/practicalgeomet00bradgoog

[5]The method is given as Exercise 299, page 125 in Charles Taylor's 1881 text *An introduction to the ancient and modern geometry of conics, being a geometrical treatise on the conic sections with a collection of problems and historical notes and prolegomena*. The text is one of Cornell Universitiy's preserved Historical Math Monographs and available at http://ebooks.library.cornell.edu/cgi/t/text/text-idx?c=math;idno=00800001
Credit is given to the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, vol. III, pp. 151, 227 (18866) in a footnote. A copy of the pertinent page from that publication is included in this document.

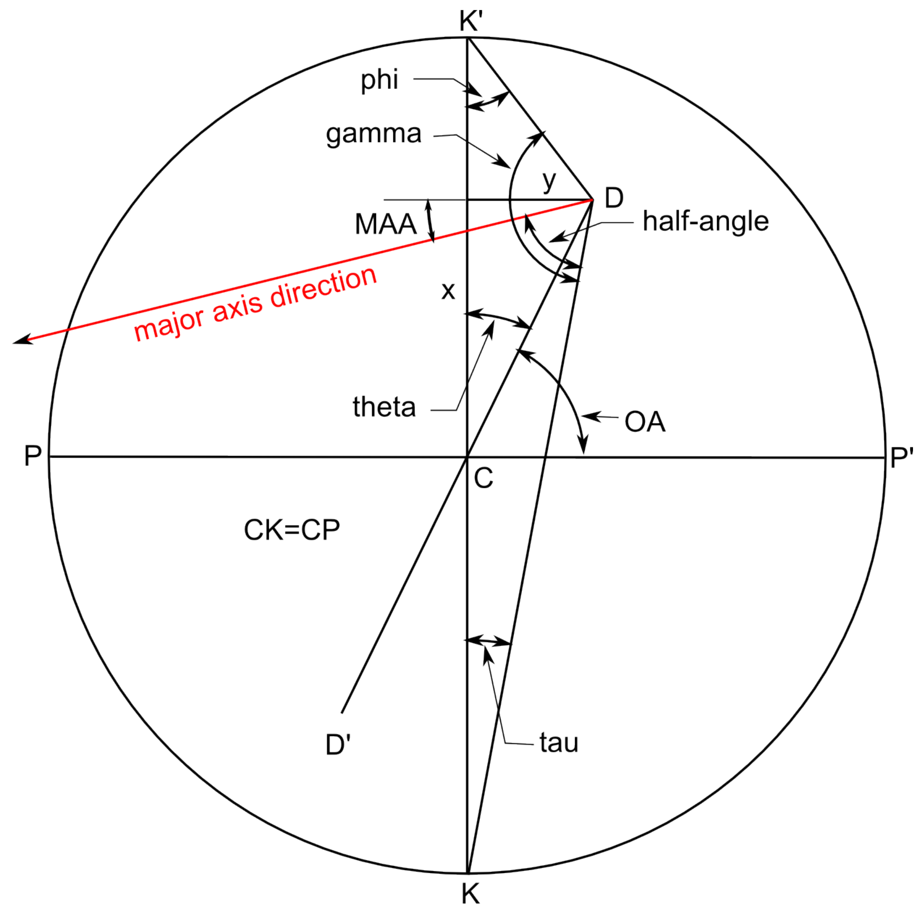


Figure 8: Taylor's Method Equation Variable Diagram

$$
\begin{aligned}
theta &= 90 - OA \\
x &= CD * COS(theta) \\
y &= x * TAN(theta) \\
phi &= ATAN2(CD * SIN(theta), (CP - x)) \\
K'D &= SQRT(CP^2 * CD^2 - 2 * CP * CD * COS(theta)) \\
KD &= SQRT(CP^2 * CD^2 - 2 * CP * CD * COS(90 + OA)) \\
MajorAxis &= KD + KD' \\
MinorAxis &+ KD - KD' \\
tau &= ATAN(y/(CP + x)) \\
HalfAngle &= (180 - (tau + phi))/2 \\
MAA &= 90 - tau - HalfAngle
\end{aligned}
$$

The next page is a copy of the page in the Oxford, Dublin and Cambridge Messenger of Mathematics referenced by Taylor and on which his Exercise 299 is based. Following that is the code for a macro implementation of three functions for getting the lengths of the Major and Minor axes and the direction of the Major axis relative to the principal diameter based on Taylor's method.

In a footnote to Exercise 299, page 125, of THE ANCIENT AND MODERN GEOMETEY OF CONICS, Charles Taylor references this article. Oxford, Cambridge, and Dublin Messenger of Mathematics}, vol. III, pp. 151 (1866);

CONSTRUCTION FOR AXES OF AN ELLIPSE.

By *R. Tucker, M.A.*

Given a pair of conjugate diameters to construct the conic.

I have not met with the following simple construction for finding the axes of a central conic, given a pair of semi-conjugate diameters, in any of the ordinary text books.

Let CP, CD (fig. 43) be the given semi-conjugate diameters, through C draw a straight line perpendicular to CP, and on it measure, from C, $CK=CP=CK'$, join KD, $K'D$, then

$$KD+K'D=\text{major-axis},$$

and

$$K'D-KD=\text{minor-axis}.$$

For, employing the usual notation, we have

$$\begin{aligned}(KD)^2 &= a'^2+b'^2-2a'b'\cos KCD\\ &= a'^2+b'^2-2a'b'\cos\left(\gamma-\frac{\pi}{2}\right)\\ &= a'^2+b'^2-2a'b'\sin\gamma\\ &= a^2+b^2-2ab\\ &= (a-b)^2,\end{aligned}$$

and therefore $KD=a-b$.

Similarly

$$\begin{aligned}(K'D)^2 &= a'^2+b'^2-2a'b'\cos\left(\frac{3\pi}{2}-\gamma\right)\\ &= a'^2+b'^2+2a'b'\sin\gamma\\ &= (a+b)^2,\end{aligned}$$

hence $K'D=a+b$,

and therefore $KD+K'D=2a=\text{major-axis}$,

$$K'D-KD=2b=\text{minor-axis}.$$

Having thus found the semi-major and semi-minor axes, we may proceed to find the direction of the axes.*

With centres P and D and distances equal $\dfrac{KD+K'D}{2}$ describe circles cutting CD, CP in E, F; also draw PT parallel to CD and DT parallel to CP, then if $\angle T'PH=\angle EPT$, PH cuts DF produced in H, one of the foci; similarly the other focus may be found, and hence the axes are determined in position.

Brighton College,
April 6, 1865.

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* The axes are parallel to the bisectors of the $\angle KDK'$.—Ed.

```
Attribute VB_Name = "DiameterToAxis"
' 8-19-2012 version
'  corrected parameter order in myAnt2() was (y,x) now (x,y)
'
' Parameters
'  CP   Principal semi-diameter
'  CD   Conjugate semi-diameter
'  OA   Ordinate Angle
'  MAA  Major Axis Angle
'  MajAxis   Major Axis
'  ConjAxis  Conjugate Axis
' Functions
'  get_MajAxis(CP, CD, OA)
'  get_ConjAxis(CP, CD, OA)
'  get_MAA(CP, CD, OA)
'  myAtn2(x,y) returns arc tangent (y/h) in range 0 to pi for postive y
'  ArcTan2 is an alternate to myAtn2
'
Function get_MajAxis(CP As Double, CD As Double, OA As Double) As Double
'  Get Major Axis length
Dim MajAxis As Double
Call do_the_math(CP, CD, OA, MajAxis)
get_MajAxis = MajAxis
End Function
'
Function get_ConjAxis(CP As Double, CD As Double, OA As Double) As Double
'  Get Conjugate Axis length
Dim ConjAxis As Double
Call do_the_math(CP, CD, OA, , ConjAxis)
get_ConjAxis = ConjAxis
End Function
'
Function get_MAA(CP As Double, CD As Double, OA As Double) As Double
'  Get Major Axis Angle relative to first parameter
'     subtract return value from ordinate angle for relative to second.
Dim MAA As Double
Call do_the_math(CP, CD, OA, , , MAA)
get_MAA = MAA
End Function
'
Private Sub do_the_math(CP As Double, CD As Double, OA As Double, Optional _
    MajAxis As Double, Optional ConjAxis As Double, Optional MAA As Double)
'  VBA uses ' to start a comment so I will append
'  an _ instead  i.e., K_D for K'D
'
Dim theta, phi, tau As Double
Dim X, Y, KD, K_D, arg As Double
Dim halfAngle As Double
With WorksheetFunction
' use with as radians and degrees not VBA functions
theta = .Radians(90 - OA)
X = CD * Cos(theta)
Y = X * Tan(theta)
phi = .Degrees(myAtn2((CP - X), CD * Sin(theta)))
```

```
'phi = .Degrees(ArcTan2((CP - X), CD * Sin(theta)))
K_D = Sqr(CP ^ 2 + CD ^ 2 - 2 * CP * CD * Cos(theta))
KD = Sqr(CP ^ 2 + CD ^ 2 - 2 * CP * CD * Cos(.Radians(90 + OA)))
MajAxis = KD + K_D
ConjAxis = KD - K_D
tau = .Degrees(Atn(Y / (CP + X)))
halfAngle = (180 - (tau + phi)) / 2
MAA = 90 - tau - halfAngle
End With
'
End Sub
'
Function myAtn2(ByVal X As Double, ByVal Y As Double) As Double
With WorksheetFunction
    On Error GoTo DivideError
    myAtn2 = Atn(Y / X)
    If (X < 0) Then
        If (Y < 0) Then myAtn2 = myAtn2 - .PI Else myAtn2 = myAtn2 + .PI
    End If
    Exit Function
DivideError:
    If Abs(Y) > Abs(X) Then   'Must be an overflow
        If Y > 0 Then myAtn2 = .PI / 2 Else myAtn2 = -.PI / 2
    Else
        myAtn2 = 0   'Must be an underflow
    End If
    Resume Next
End With
End Function
'
Function ArcTan2(X As Double, Y As Double) As Double
Const PI As Double = 3.14159265358979
Const PI_2 As Double = 1.5707963267949
    Select Case X
        Case Is > 0
            ArcTan2 = Atn(Y / X)
        Case Is < 0
            ArcTan2 = Atn(Y / X) + PI * Sgn(Y)
            If Y = 0 Then ArcTan2 = ArcTan2 + PI
        Case Is = 0
            ArcTan2 = PI_2 * Sgn(Y)
    End Select
End Function
```